

ON THE VANISHING OF RELATIVE NEGATIVE K-THEORY

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ABSTRACT. In this article, we study the relative negative K-groups $K_{-n}(f)$ of a map $f : X \rightarrow S$ of schemes. We prove a relative version of the Weibel conjecture i.e. if $f : X \rightarrow S$ is a smooth affine map of noetherian schemes with $\dim S = d$ then $K_{-n}(f) = 0$ for $n > d + 1$ and the natural map $K_{-n}(f) \rightarrow K_{-n}(f \times \mathbb{A}^r)$ is an isomorphism for all $r > 0$ and $n > d$. We also prove a vanishing result for relative negative K-groups of a subintegral map.

1. INTRODUCTION

In 1980, Weibel conjectured that for a d -dimensional noetherian scheme X , the negative K -groups should vanish after the dimension and the natural map $K_{-d}(X) \rightarrow K_{-d}(X \times \mathbb{A}^r)$ for all $r > 0$ should be an isomorphism i.e. X should be K_{-d} -regular (see Question 2.9 of [21]). Significant progress related to this conjecture has been made in the articles [3], [4], [5], [6], [7], [9], [10], [23] by various authors. Very recently, a complete answer is given in [8] by Kerz-Strunk-Tamme.

Let $f : X \rightarrow S$ be a morphism of schemes. By definition, the n -th relative K-group $K_n(f)$ is $\pi_n K(f)$, where $n \in \mathbb{Z}$ and $K(f)$ is the homotopy fiber of $K(S) \rightarrow K(X)$. Here and throughout, $K(X)$ denotes the non-connective Bass K -theory spectrum of the scheme X . Similarly, by replacing K by KH , we get the n -th relative homotopy K -group $KH_n(f)$. We say that $f : X \rightarrow S$ is K_n -regular if the natural map $K_n(f) \rightarrow K_n(f \times \mathbb{A}^r)$ is an isomorphism for all $r > 0$. In this article, we are considering Weibel conjecture in the relative setting. More precisely, we are interested in investigating the condition on f under which an analogous vanishing and regularity result holds for the relative negative K -groups.

Firstly, we consider the case when f is a smooth affine map. We discuss such a case in Section 3. Using the technique of [7] and [8], we prove the following

Theorem 1.1. *Let $f : X \rightarrow S$ be a smooth, affine map of noetherian schemes. Assume that $\dim S = d$. Then $K_{-n}(f) = 0$ for $n > d + 1$ and f is K_{-n} -regular for $n > d$.*

Date: February 22, 2017.

1991 Mathematics Subject Classification. 14C35, 19D35, 19E08.

Key words and phrases. Relative negative K-groups, Smooth map, Subintegral map.

Author was supported by TIFR, Mumbai Postdoctoral Fellowship.

Secondly, we consider the case when f is smooth, but may not be affine. In this situation, we are able to prove a vanishing result for relative negative homotopy K -groups assuming the resolution of singularities. We prove such a result in Section 4. In Section 4, all the schemes are defined over a field k and we assume that the resolution of singularities holds over k . Here is our result

Theorem 1.2. *Let $f : X \rightarrow S$ be a smooth and surjective map of noetherian schemes over a field k . Assume that $\dim S = d$. Then $KH_{-n}(f) = 0$ for $n > d + 1$ and $H_{cdh}^d(S, \mathcal{K}_{-1, cdh}^f) = KH_{-d-1}(f)$. Here $\mathcal{K}_{n, cdh}^f$ is the cdh -sheafification of the presheaf $U \mapsto K_n(U, f^{-1}U)$.*

However, we notice that the surjectivity of f can be dropped in the above result when f is an étale map (see Remark 4.8 and Theorem 4.9). Using the Theorem 1.2, we show that $KH_{-n}(\mathbb{P}_X^t) = 0$ for $t \geq 0$ and $n > d$, where X is a d -dimensional noetherian scheme over k (see Corollary 4.6).

Next, we discuss the situation when the map $f : X \rightarrow S$ may not be smooth. In particular, we consider subintegral maps. In [17], the author and Weibel have shown that if $f : X = \text{Spec}(B) \rightarrow S = \text{Spec}(A)$ is a subintegral map (i.e. $A \hookrightarrow B$ is subintegral) then $K_{-n}(f) = 0$ for $n > 0$ (see Proposition 2.5 of [17]). It has also been observed in [17, Example 6.6] that if S is not affine then the above mentioned result may fail. For example, consider $S = \mathbb{P}_k^1$ and $X = \text{Spec}(\mathcal{O}_B)$ where $\mathcal{O}_B = \mathcal{O}_S \oplus \mathcal{O}(-2)$ with $\mathcal{O}(-2)$ is a square zero ideal. In this situation, $K_{-1}(f) \neq 0$. This suggests that the relative negative K -groups may be nonzero at the dimension (i.e. $K_{-\dim S}$) in the non affine situation. So it is natural to wonder what the groups $K_{-n}(f)$ are for subintegral morphisms with non affine base. This is answered in Section 5 by proving the following theorem.

Theorem 1.3. *Let $f : X \rightarrow S$ be a subintegral morphism of noetherian schemes. Assume that $\dim S = d$. Then*

- (1) $K_{-n}(f) = 0$ for $n > d$,
- (2) $H_{zar}^d(S, f_* \mathcal{O}_X^\times / \mathcal{O}_S^\times) = K_{-d}(f)$,
- (3) *If X and S are \mathbb{Q} -schemes then $H_{et}^d(S, f_* \mathcal{O}_X^\times / \mathcal{O}_S^\times) = K_{-d}(f)$.*

As a corollary, we obtain $K_{-n}(X) \cong K_{-n}(X_{\text{sn}})$ for $n > d$ and $K_{-d}(X) \rightarrow K_{-d}(X_{\text{sn}})$ is surjective, where X is a d -dimensional noetherian scheme and X_{sn} is the seminormalization of X (see Corollary 5.7).

In Section 6, we prove a relative version of Vorst regularity result i.e. K_n -regularity implies K_{n-1} -regularity. More precisely, we prove

Theorem 1.4. *If $f : X = \text{Spec}(B) \rightarrow S = \text{Spec}(A)$ is K_n -regular then f is K_{n-1} -regular.*

As a consequence, we show that if $f : A \hookrightarrow B$ is subintegral ring extension then f can not be K_n -regular and $K_n(f) \neq 0$ for $n \geq 0$ (see Proposition 6.1).

Finally, we conclude this article with the following theorem (see Section 7).

Theorem 1.5. *Let S be a noetherian scheme of dimension d . The the following are equivalent*

- (1) $K_{-n}(f) = 0$ for $n > d + 1$ and f is K_{-n} -regular for $n > d$, for every smooth affine map $f : X \rightarrow S$ of noetherian schemes.
- (2) (Weibel Conjecture) $K_{-n}(S) = 0$ for $n > d$ and S is K_{-n} -regular for $n \geq d$.

Acknowledgements: The author is grateful to Charles Weibel for his valuable comments and various suggestions during the preparation of this article. He would also like to thank Omprokash Das for some useful discussions.

2. PRELIMINARIES

Subintegral and Seminormal extension. Let $A \hookrightarrow B$ be a commutative ring extension. This extension $A \hookrightarrow B$ is subintegral if B is integral over A and $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is a bijection inducing isomorphisms on all residue fields. We say that $A \hookrightarrow B$ is seminormal (or A is seminormal in B) if whenever $b \in B$ and $b^2, b^3 \in A$ then $b \in A$. More details can be found in [13], [18].

Relative K-groups. Given a map $f : X \rightarrow S$ of schemes, $K_n(f) = \pi_n K(f)$, where $K(f)$ is the homotopy fiber of $K(S) \rightarrow K(X)$. These relative K-groups fit into the following exact sequence $\text{Seq}(K_n, f)$

$$(2.1) \quad \cdots \rightarrow K_n(f) \rightarrow K_n(S) \rightarrow K_n(X) \rightarrow K_{n-1}(f) \rightarrow K_{n-1}(S) \rightarrow \cdots$$

For details see [2], [24].

Relative Picard groups. We also have a notion of relative Picard group $\text{Pic}(f)$ for a map $f : X \rightarrow S$ of schemes. The relative $\text{Pic}(f)$ is the abelian group generated by $[L_1, \alpha, L_2]$, where the L_i are line bundles on S and $\alpha : f^*L_1 \rightarrow f^*L_2$ is an isomorphism. The relations are:

- (1) $[L_1, \alpha, L_2] + [L'_1, \alpha', L'_2] = [L_1 \otimes L'_1, \alpha \otimes \alpha', L_2 \otimes L'_2]$;
- (2) $[L_1, \alpha, L_2] + [L_2, \beta, L_3] = [L_1, \beta\alpha, L_3]$;
- (3) $[L_1, \alpha, L_2] = 0$ if $\alpha = f^*(\alpha_0)$ for some $\alpha_0 : L_1 \cong L_2$.

This relative Picard group $\text{Pic}(f)$ fits into the following exact sequence

$$(2.2) \quad \mathcal{O}^\times(S) \rightarrow \mathcal{O}^\times(X) \rightarrow \text{Pic}(f) \rightarrow \text{Pic}(S) \rightarrow \text{Pic}(X).$$

Some relevant details and basic properties can be found in [1], [16], [17].

Relative Homotopy K-groups. Let $\Delta^n = \text{Spec}(\mathbb{Z}[t_1, t_2, \dots, t_n]/(t_1 + t_2 + \dots + t_n - 1))$. Then the n -th homotopy K-group of a scheme X is $KH_n(X) = \pi_n(KH(X))$, where $n \in \mathbb{Z}$ and $KH(X) = \text{hocolim}_j K(X \times \Delta^j)$. For a map of schemes $f : X \rightarrow S$, let $KH(f)$ be the homotopy fiber of $KH(S) \rightarrow KH(X)$. In fact, $KH(f) = \text{hocolim}_j K(f \times \Delta^j)$ by Lemma 5.19 of [19]. Then for $n \in \mathbb{Z}$, the n -th relative homotopy K-group of f is $KH_n(f) = \pi_n(KH(f))$. The relative homotopy K-groups fit into the following exact sequence $\text{Seq}(KH_n, f)$

$$(2.3) \quad \dots \rightarrow KH_n(f) \rightarrow KH_n(S) \rightarrow KH_n(X) \rightarrow KH_{n-1}(f) \rightarrow KH_{n-1}(S) \rightarrow \dots$$

For more details, we refer [22] and Chapter IV.12 of [24].

Remark 2.1. For a scheme X , there is a natural map $K(X) \rightarrow KH(X)$. Therefore, we get a natural map $K(f) \rightarrow KH(f)$ for any map $f : X \rightarrow S$ of schemes. In particular, there are natural maps $K_n(f) \rightarrow KH_n(f)$ for all n . For every scheme X , $KH_n(X) \cong KH_n(X \times \mathbb{A}^t)$ for all n and $t \geq 0$. It is also well known that for a regular scheme X , $K_n(X) \cong KH_n(X)$ for all n . Using the exact sequences (2.1) and (2.3), the following facts are easy to check

- (1) If X and S are regular schemes then $K_n(f) \cong KH_n(f)$ for all n .
- (2) (Homotopy Invariance) $KH_n(f) \cong KH_n(f \times \mathbb{A}^t)$ for all n .

3. RELATIVE NEGATIVE K-THEORY OF SMOOTH, AFFINE MAPS

In this section, we prove Theorem 1.1, which is a vanishing and regularity result for relative negative K -groups of a smooth, affine map. To prove this, we need some preparations. Let us begin with the following observation.

Lemma 3.1. *Let $f : X \rightarrow S$ be a map of noetherian schemes with $\dim S = d$. Then the following are true:*

- (1) *for $n > d$, $K_{-n}(X) = 0$ if and only if $K_{-n-1}(f) = 0$.*
- (2) *for $n \geq d$, X is K_{-n} -regular if and only if f is K_{-n-1} -regular.*

Proof. By Theorem B of [8], $K_{-n}(S) = 0$ for $n > d$ and S is K_{-n} -regular for $n \geq d$. Now the first assertion follows from the long exact sequence (2.1). For the second assertion, apply N^i to the sequence (2.1) and use the fact S is K_{-n} -regular for $n \geq d$. \square

Lemma 3.2. *Let $f : X \rightarrow S$ be a smooth map of noetherian schemes with $\dim S = 0$. Assume that S is reduced. Then $K_{-n}(f) = 0$ for $n > 1$ and f is K_{-n} -regular for $n > 0$.*

Proof. First we claim that $S = \{s_1, s_2, \dots, s_n\}$, where $s_i = \text{Spec}(k_i)$ with k_i a field. Let T be an irreducible component of S . The topological space S carries the discrete topology because it is a zero dimensional noetherian space. So, T is open in S and $\dim T = 0$.

Since S is reduced, T is also reduced. Then T is a zero dimensional noetherian integral scheme. We know that every zero dimensional noetherian scheme is a disjoint union of spectra of artinian local rings. Therefore, $T = \text{Spec}(A)$, where A is an artinian domain. Since S has a finite number of irreducible components and every artinian domain is a field, we get the claim.

Since f is smooth, each fiber $f^{-1}(s) = X_s$ is regular for $s \in S$. By the above claim, $S = \text{Spec}(k_1) \sqcup \cdots \sqcup \text{Spec}(k_n)$ and for each i , $\text{Spec}(k_i) \rightarrow S$ is an open immersion. Then $X_{s_i} \rightarrow X$ is also an open immersion for each i . Now, we can write X as a finite disjoint union of open subschemes X_{s_i} which are regular. Hence X is regular. Then X is K_n -regular for all n and $K_{-n}(X) = 0$ for $n > 0$. We also have $K_{-n}(S) = 0$ for $n > 0$ and S is K_{-n} -regular for $n \geq 0$. Therefore by (2.1), $K_{-n}(f) = 0$ for $n > 1$ and f is K_{-n} -regular for $n > 0$. \square

For a morphism of schemes $f : X \rightarrow S$, let $\mathcal{K}(f)$ be the presheaf of spectra on S , defined as

$$\mathcal{K}(f)(U) = \text{hofib}[\mathcal{K}(S)(U) \rightarrow \mathcal{K}(X)(X \times_S U)],$$

where $\mathcal{K}(X)$ is the presheaf of spectra on X (resp. $\mathcal{K}(S)$ on S). Similarly, we can define the nil presheaf of spectra $N^i \mathcal{K}(f)$ on S for $i > 0$.

Lemma 3.3. $\mathcal{K}(f)$ and $N^i \mathcal{K}(f)$ satisfy Zariski descent.

Proof. We have a sequence of presheaves of spectra on S ,

$$\mathcal{K}(f) \rightarrow \mathcal{K}(S) \rightarrow f_* \mathcal{K}(X).$$

It is easy to check that if $\mathcal{K}(X)$ satisfies Zariski descent then $f_* \mathcal{K}(X)$ does too. By Corollary V.7.10 of [24], $\mathcal{K}(X)$ satisfies Zariski descent. Then $\mathcal{K}(f)$ satisfies Zariski descent (see Exercise V.10.1 of [24]). By a similar argument, $N^i \mathcal{K}(f)$ satisfies Zariski descent. \square

For a morphism of schemes $f : X \rightarrow S$, let \mathcal{K}_n^f be the Zariski sheafification of the presheaf $U \mapsto K_n(U, f^{-1}U)$.

Lemma 3.4. Let $f : X \rightarrow S$ be an affine map of noetherian schemes with $\dim S = d$. Then the canonical map $K_{-n}(f) \rightarrow K_{-n}(f \times_S S_{\text{red}})$ is an isomorphism for $n > d$, where $f \times_S S_{\text{red}} : X \times_S S_{\text{red}} \rightarrow S_{\text{red}}$.

Proof. Write \tilde{f} for $f \times_S S_{\text{red}}$. First we suppose that $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$. Then $X \times_S S_{\text{red}} = \text{Spec}(B \otimes_A A/\text{nil}(A))$. Note that $\text{nil}(A)B$ is a nil ideal of B . Then by comparing sequences (see (2.1)) $\text{Seq}(K_n, f)$ and $\text{Seq}(K_n, \tilde{f})$, we get $K_{-n}(f) \cong K_{-n}(\tilde{f})$ for $n > 0$ because for any ring R , $K_{-n}(R) \cong K_{-n}(R/I)$ for $n \geq 0$ with I a nil ideal. Now by looking at the stalk level it is easy to see that $\mathcal{K}_n^f \cong \mathcal{K}_n^{\tilde{f}}$ for all $n < 0$ as a Zariski sheaf

on S . There is a canonical map of Zariski descent spectral sequence for S (Theorem 10.3 of [20]),

$$E_2^{p,q} = H^p(S, \mathcal{K}_{-q}^f) \Rightarrow K_{-p-q}(f)$$

to for S_{red}

$$E_2^{p,q} = H^p(S_{red}, \mathcal{K}_{-q}^{\tilde{f}}) \Rightarrow K_{-p-q}(\tilde{f}),$$

which is an isomorphism on $E_2^{p,q}$ page for $q > 0$. Moreover, Zariski cohomological dimension is at most d . Hence the result. \square

Lemma 3.5. *Let $f : X \rightarrow S$ be a map of noetherian schemes. Suppose $\dim S = d$. Write f_s for the map $X \times_S \mathcal{O}_{S,s} \rightarrow \mathcal{O}_{S,s}$, $s \in S$. Then the following are true:*

- (1) *If $K_{-n}(f_s) = 0$ for all $s \in S$ with $n > \dim \mathcal{O}_{S,s} + 1$ then $K_{-n}(f) = 0$ for $n > d + 1$.*
- (2) *If $N^i K_{-n}(f_s) = 0$ for all $s \in S$ with $n > \dim \mathcal{O}_{S,s}$ and $i > 0$ then $N^i K_{-n}(f) = 0$ for $n > d$ and $i > 0$.*

Proof. The result is clear by Lemma 3.3 and Proposition 6.1 of [8]. More precisely, apply Proposition 6.1 of [8] to the presheaves of spectra $\mathcal{K}(f)[-1]$ and $N^i \mathcal{K}(f)$. \square

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: By Lemma 3.5, we can assume that S is affine. We can also assume that S is reduced by Lemma 3.4. We prove by using induction on $\dim S$. If $\dim S = 0$ then the assertion is clear by Lemma 3.2. Suppose $d > 0$. Assume that for every smooth, affine map $X \rightarrow S$ with $\dim S < d$, we have $K_{-n}(X) = 0$ for $n > \dim S$ (see Lemma 3.1). Let $i < -d$ and consider an element ξ in $K_i(X)$. Here f is smooth and quasi-projective. Apply Proposition 5 of [7] to the map $f : X \rightarrow S$. Then there exist a projective birational map $p : S' \rightarrow S$ such that $\tilde{p}^* \xi = 0$ where $\tilde{p} : X' = X \times_S S' \rightarrow X$. We can choose a nowhere dense closed subset $Y \hookrightarrow S$ such that p is an isomorphism outside Y . Then we obtain the following abstract blow-up squares

$$\begin{array}{ccc} Y' & \longrightarrow & S' \\ \downarrow & & \downarrow p \\ Y & \longrightarrow & S \end{array} \quad \text{and} \quad \begin{array}{ccc} X \times_S Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \tilde{p} \\ X \times_S Y & \longrightarrow & X \end{array}$$

By applying Theorem A of [8], we get a long exact sequence

$$\cdots \rightarrow \varprojlim_n K_{i+1}(X \times_S Y'_n) \rightarrow K_i(X) \rightarrow \varprojlim_n K_i(X \times_S Y_n) \oplus K_i(X') \rightarrow \cdots$$

of pro-groups. Here Y_n (resp. Y'_n) is the n -th infinitesimal thickening of Y (resp. Y') in S (resp. S'). Observe that $X \times_S Y \rightarrow Y$ and $X \times_S Y' \rightarrow Y$ are smooth, affine with $\dim Y < d$ and $\dim Y' < d$. Then by induction hypothesis, the pro-groups involving

$K_i(X \times_S Y'_n)$ and $K_i(X \times_S Y_n)$ vanish. Therefore, $\tilde{p}^* : K_i(X) \rightarrow K_i(X')$ is injective and hence $\xi = 0$. This proves the first part.

In the second part, we can assume that S is affine by Lemma 3.5. Then X is affine. Now by the proof of Lemma 3.4, we can also assume that S is reduced. Again, we use induction on the dimension of S . If $\dim S = 0$ then the assertion is clear by Lemma 3.2. Suppose $d > 0$. Assume that for every smooth, affine map $X \rightarrow S$ with $\dim S < d$, we have $N^i K_{-n}(X) = 0$ for $n \geq \dim S$ and $i > 0$ (see Lemma 3.1). Let $n \geq d$ and $r > 0$. For each r , we can argue the inductive step separately. Consider $\xi \in K_{-n}(\mathbb{A}_X^r)$. Apply Proposition 5 of [7], to the map $\mathbb{A}_X^r \rightarrow \mathbb{A}_S^r \rightarrow S$, which is smooth and quasi-projective. Then there exist a projective birational map $p : S' \rightarrow S$ such that $\tilde{p}^* \xi = 0$ where $\tilde{p} : \mathbb{A}_{X'}^r \rightarrow \mathbb{A}_X^r$ and $X' = X \times_S S'$. We can choose a nowhere dense closed subset $Y \hookrightarrow S$ such that p is an isomorphism outside Y . Now we have the following commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow & \text{“}\lim_n\text{” } K_{i+1}(X \times_S Y'_n) & \longrightarrow & K_i(X) & \longrightarrow & \text{“}\lim_n\text{” } K_i(X \times_S Y_n) \oplus K_i(X') & \longrightarrow \dots \\
& \beta_{Y'}^* \downarrow & & \beta^* \downarrow & & \beta_Y^* \oplus \beta_{X'}^* \downarrow & \\
\longrightarrow & \text{“}\lim_n\text{” } K_{i+1}(\mathbb{A}_{X \times_S Y'_n}^r) & \longrightarrow & K_i(\mathbb{A}_X^r) & \longrightarrow & \text{“}\lim_n\text{” } K_i(\mathbb{A}_{X \times_S Y_n}^r) \oplus K_i(\mathbb{A}_{X'}^r) & \longrightarrow \dots,
\end{array}$$

where the horizontal sequence is exact by Theorem A of [8]. Here β is the projection map $\mathbb{A}_X^r \rightarrow X$ and β^* is the induced morphism. Since $\dim Y < d$ and $\dim Y' < d$, β_Y^* , $\beta_{Y'}^*$ are isomorphism by induction hypothesis. By the first part the pro-groups in the upper horizontal sequence involving $K_i(X \times_S Y_n)$ vanishes. Now a simple diagram chase gives that β^* is surjective. Since β^* is always injective, we get the result. \square

4. RELATIVE NEGATIVE HOMOTOPY K-THEORY OF SMOOTH, SURJECTIVE MAPS

In this section, all the schemes are defined over a field k and we assume that the resolution of singularities holds over k . The main goal is to prove Theorem 1.2, which is a vanishing result for relative negative homotopy K-groups of a smooth, surjective map.

Lemma 4.1. *Let $f : X \rightarrow S$ be a map of schemes over a field k with S smooth. Suppose f factors into $X \xrightarrow{g} \mathbb{A}_S^t \rightarrow S$ with g étale. Then $K_{-n}(f) \cong K_{-n}(g)$ for all n and $K_{-n}(g) = 0$ for $n > 1$.*

Proof. Since S is smooth, $K_n(\mathbb{A}_S^t) \cong K_n(S)$ by K_n -regularity for all n . Now by comparing the exact sequence (2.1) for the maps f and g , we get the first assertion. Note that \mathbb{A}_S^t is regular. Then X is regular by Proposition I.3.17(c) of [11], because g is étale. It is well known that the negative absolute K-theory of regular scheme vanish. Hence the second assertion by the exact sequence (2.1). \square

Lemma 4.2. *Let $f : X \rightarrow S$ be a map of schemes. Let $x \in X$. Let $V \subset S$ be an affine open nbd of $f(x)$. If f is smooth at x , then there exists an integer $d \geq 0$ and affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that there exists a commutative diagram*

$$\begin{array}{ccc} X & \leftarrow & U \xrightarrow{\pi} \mathbb{A}_V^d \\ \downarrow & & \downarrow \swarrow \\ Y & \leftarrow & V, \end{array}$$

where π is étale.

Proof. See Lemma 34.20 of [14]. □

Let $\mathcal{K}_{n,cdh}^f$ be the cdh-sheafification of the presheaf $U \mapsto K_n(U, f^{-1}U)$. By replacing K by KH , we get $\mathcal{KH}_{n,cdh}^f$.

Lemma 4.3. *Let $f : X \rightarrow S$ be a smooth and surjective map of schemes over a field k . Then $\mathcal{KH}_{n,cdh}^f \cong \mathcal{K}_{n,cdh}^f$ as a cdh sheaf on S . Moreover, $\mathcal{K}_{n,cdh}^f$ is zero for $n < -1$.*

Proof. Pick any $s \in S$. Since f is surjective, $f(x) = s$ for some $x \in X$. Let $V \subset S$ be an affine open nbd of $f(x)$. Now by Lemma 4.2, there exists an integer $d \geq 0$ and affine open $U \subset X$ with $x \in U$ and $f(U) \subset V$ such that locally f factor as $U = X \times_S V \xrightarrow{et} \mathbb{A}_V^d \rightarrow V$. In cdh topology, schemes are locally smooth. So at stalk, $\mathcal{K}_{n,cdh}^f$ (resp. $\mathcal{KH}_{n,cdh}^f$) is $K_n(X \times_S R \xrightarrow{et} \mathbb{A}_R^d \rightarrow R)$, (resp. KH_n) where R is a regular local ring. By Proposition I.3.17(c) of [11], $X \times_S R$ is regular because \mathbb{A}_R^d is regular. Then we get $K_n(X \times_S R \xrightarrow{et} \mathbb{A}_R^d \rightarrow R) \cong KH_n(X \times_S R \xrightarrow{et} \mathbb{A}_R^d \rightarrow R)$ by Remark 2.1. Therefore, at stalk level \mathcal{K}_n^f and \mathcal{KH}_n^f are isomorphic. Also, $K_n(X \times_S R \xrightarrow{et} \mathbb{A}_R^d) = 0$ for $n < -1$ by Lemma 4.1. Hence the assertion. □

For a morphism of schemes $f : X \rightarrow S$, let $\mathcal{KH}(f)$ be the presheaf of spectra on S , defined as

$$\mathcal{KH}(f)(U) = \text{hofib}[\mathcal{KH}(S)(U) \rightarrow \mathcal{KH}(X)(X \times_S U)],$$

where $\mathcal{KH}(X)$ is the presheaf of spectra on X (resp. $\mathcal{KH}(S)$ on S).

Lemma 4.4. *$\mathcal{KH}(f)$ satisfies cdh descent.*

Proof. By Theorem 3.9 of [3], $\mathcal{KH}(X)$ satisfies cdh descent. The rest of the arguments are similar to Lemma 3.3. □

Proof of Theorem 1.2: By Lemma 4.4, $\mathcal{KH}(f)$ satisfies cdh descent. Therefore, we have a descent spectral sequence (by Theorem 3.4 of [4] and Theorem 2.8 of [6])

$$E_2^{p,q} = H_{cdh}^p(S, \mathcal{KH}_{-q,cdh}^f) \Rightarrow KH_{-p-q}(f).$$

Also by Lemma 4.3, $\mathcal{KH}_{q,cdh}^f \cong \mathcal{K}_{q,cdh}^f$ as a cdh sheaf on S and for $q < -1$, $\mathcal{K}_{q,cdh}^f$ is zero. Moreover, the cdh cohomological dimension is at most d . Hence, we get $KH_{-n}(f) = 0$ for $n > d + 1$ and $H_{cdh}^d(S, \mathcal{K}_{-1,cdh}^f) = KH_{-d-1}(f)$. □

Some explicit calculations in lower dimensions are given in the following corollary. Write \mathcal{K}_q^f for $\mathcal{K}_{q,cdh}^f$.

Corollary 4.5. *Let f be as in Theorem 1.2. Then*

- (1) *If $\dim S = 0$ then $KH_{-1}(f) \cong H_{cdh}^0(S, \mathcal{K}_{-1}^f)$ and $KH_{-n}(f) = 0$ for $n > 1$.*
- (2) *If $\dim S = 1$ then the sequence*

$$0 \rightarrow H_{cdh}^1(S, \mathcal{K}_0^f) \rightarrow KH_{-1}(f) \rightarrow H_{cdh}^0(S, \mathcal{K}_{-1}^f) \rightarrow 0$$

is exact, $KH_{-2}(f) \cong H_{cdh}^1(S, \mathcal{K}_{-1}^f)$ and $KH_{-n}(f) = 0$ for $n > 2$.

- (3) *If $\dim S = 2$ then the sequence*

$$0 \rightarrow H_{cdh}^1(S, \mathcal{K}_0^f) \rightarrow KH_{-1}(f) \rightarrow H_{cdh}^0(S, \mathcal{K}_{-1}^f) \rightarrow H_{cdh}^2(S, \mathcal{K}_0^f) \rightarrow KH_{-2}(f) \rightarrow H_{cdh}^1(S, \mathcal{K}_{-1}^f) \rightarrow 0$$

is exact, $KH_{-3}(f) \cong H_{cdh}^2(S, \mathcal{K}_{-1}^f)$ and $KH_{-n}(f) = 0$ for $n > 3$.

Proof. The assertions are clear from the following seven term exact sequence

$$\begin{aligned} 0 \rightarrow H_{cdh}^1(S, \mathcal{K}_0^f) \rightarrow KH_{-1}(f) \rightarrow H_{cdh}^0(S, \mathcal{K}_{-1}^f) \rightarrow H_{cdh}^2(S, \mathcal{K}_0^f) \\ \rightarrow \ker[KH_{-2}(f) \rightarrow H_{cdh}^0(S, \mathcal{K}_{-2}^f)] \rightarrow H_{cdh}^1(S, \mathcal{K}_{-1}^f) \rightarrow H_{cdh}^3(S, \mathcal{K}_0^f). \end{aligned}$$

□

The next result is well known, but we are including it here as an application of Theorem 1.2.

Corollary 4.6. *Let X be a d -dimensional noetherian scheme over a field k . Then for all $t \geq 0$, $KH_{-n}(\mathbb{P}_X^t) = 0$ for $n > d$.*

Proof. Note that the projection $\pi : \mathbb{P}_X^t \rightarrow X$ is a smooth and surjective map. By Theorem 1.2, $KH_{-n}(\pi) = 0$ for $n > d+1$. We have $KH_{-n}(X) = 0$ for $n > d$ by Theorem 1 of [7]. Then the exact sequence (2.3) implies that $KH_{-n}(\mathbb{P}_X^t) = 0$ for $n > d$. □

Remark 4.7. The vanishing of $KH_{-n}(f)$ remains valid after the finite base change. More precisely, let $f : X \rightarrow S$ be as in Theorem 1.2. Let $h : S' \rightarrow S$ be a finite map. Then $\dim S' \leq d$. Since smooth and surjective maps are stable under base change, $f' : X \times_S S' \rightarrow S'$ is smooth and surjective. Hence $KH_{-n}(f') = 0$ for $n > d+1$.

Remark 4.8. We do not know whether Lemma 4.3 is true without surjective assumption on f . But, if $f : X \rightarrow S$ is just an étale map, then the Lemma 4.3 holds without f being surjective. Indeed, at stalk $\mathcal{KH}_{q,cdh}^f$ is $K_q(X \times_S R \xrightarrow{et} R)$, where R is a regular local ring. Then $X \times_S R$ is regular by Proposition I.3.17(c) of [11]. By Remark 2.1, at stalk level \mathcal{K}_q^f and \mathcal{KH}_q^f are isomorphic and for $q < -1$, $\mathcal{K}_{q,cdh}^f$ is zero. Therefore, the Theorem 1.2 is also true for an étale map.

In view of above remark, we are in situation to write the following

Theorem 4.9. *Let $f : X \rightarrow S$ be an étale map of noetherian schemes over a field k . Assume that $\dim S = d$. Then $KH_{-n}(f) = 0$ for $n > d + 1$ and $H_{cdh}^d(S, \mathcal{K}_{-1, cdh}^f) = KH_{-d-1}(f)$.*

5. RELATIVE NEGATIVE K-THEORY OF SUBINTEGRAL MAPS

In this section, we study the relative negative K -groups of a subintegral map of schemes. In particular, we prove Theorem 1.3. We begin with the following definition.

Definition 5.1. *Let $f : X \rightarrow S$ be a faithful affine morphism, i.e., affine and the structure map $\mathcal{O}_S \rightarrow f_*\mathcal{O}_X$ is injective. We say that f is subintegral if $\mathcal{O}_S(U) \rightarrow f_*\mathcal{O}_X(U)$ is subintegral for all affine open subsets U of S .*

Let $A \hookrightarrow B$ be a ring extension. The Roberts-Singh group $\mathcal{I}(A, B)$ (or $\mathcal{I}(f)$, where $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$) is the multiplicative group of invertible A -submodules of B . We refer section 2 of [13] for details. There is a natural group homomorphism $\psi : \mathcal{I}(f) \rightarrow \text{Pic}(f)$, $I \mapsto [I, \alpha, A]$, where $\alpha : I \otimes_A B \cong B$.

If $\mathbb{Q} \subset A$ then a natural group homomorphism $\xi_{B/A} : B/A \rightarrow \mathcal{I}(f)$ is constructed in [13] as follows: for $b \in B$, let $I_{B/A}(b) = B[t] \cap A[[t]]e^{bt}$, where t is an indeterminate. By Theorem 4.17 and Corollary 4.3 of [13], $I_{B/A}(b) \in \mathcal{I}(f[t])$. Here $f[t] : \text{Spec}(B[t]) \rightarrow \text{Spec}(A[t])$. Let $\tau : \mathcal{I}(f[t]) \xrightarrow{t \mapsto 1} \mathcal{I}(f)$. Then the homomorphism $\xi_{B/A}$ is given by $\xi_{B/A}(\bar{b}) = \tau(I_{B/A}(b))$, where $\bar{b} \in B/A$ with representative $b \in B$.

Lemma 5.2. *Let $f : A \hookrightarrow B$ be a subintegral extension of \mathbb{Q} -algebras. Then the natural map $\psi \circ \xi_{B/A} : B/A \rightarrow \text{Pic}(f)$ is an isomorphism.*

Proof. By Lemma 1.2 of [17], ψ is an isomorphism. The isomorphism of $\xi_{B/A}$ was proven in Theorem 5.6 of [13] and Theorem 2.3 of [12]. Hence the lemma. \square

The following Proposition generalizes the above result for schemes.

Proposition 5.3. *Let $f : X \rightarrow S$ be a subintegral morphism of \mathbb{Q} -schemes. Then $\text{Pic}(f) \cong H_{\text{zar}}^0(S, f_*\mathcal{O}_X/\mathcal{O}_S)$.*

Proof. Let $s \in S$. Then $(f_*\mathcal{O}_X/\mathcal{O}_S)_s \cong B_s/A_s \cong \mathcal{I}(A_s, B_s) \cong (f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)_s$, where the second isomorphism by Lemma 5.2 and the third isomorphism by the exact sequence (2.2). This implies that $f_*\mathcal{O}_X/\mathcal{O}_S \cong f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ as a sheaves on S . Now the result follows from Lemma 5.4 of [16]. \square

Proposition 5.4. *Let $f : X \rightarrow S$ be a subintegral morphism of noetherian \mathbb{Q} -schemes. Then the following are true:*

- (1) $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ is a quasi-coherent sheaf.
- (2) $H_{zar}^i(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) = H_{et}^i(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)$.
- (3) If S is affine and f is finite then for $i > 1$, $H_\tau^i(S, \mathcal{O}_S^\times) \cong H_\tau^i(X, \mathcal{O}_X^\times)$, where $\tau = \{zar, et\}$.

Proof. (1) Since f is affine, $f_*\mathcal{O}_X$ is quasi-coherent. Then the quotient $f_*\mathcal{O}_X/\mathcal{O}_S$ is also quasi-coherent. Therefore, we get the result by using the fact that $f_*\mathcal{O}_X/\mathcal{O}_S \cong f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ (see the proof of Proposition 5.3).

(2) This follows from the fact that Zariski and étale cohomology coincides for a quasi-coherent sheaf (see Remark 3.8 of [11]).

(3) Consider the long exact cohomology sequence associated to the following exact sequence of sheaves on S ,

$$1 \rightarrow \mathcal{O}_S^\times \rightarrow f_*\mathcal{O}_X^\times \rightarrow f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times \rightarrow 1.$$

Since f is finite, $H_\tau^i(S, f_*\mathcal{O}_X^\times) \cong H_\tau^i(X, \mathcal{O}_X^\times)$. By (1), $H_{zar}^i(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) = 0$ for $i > 0$, because S is affine. Hence the assertion. \square

Remark 5.5. The statement (3) of the above Proposition may fail for $i = 1$. For example, consider $A = \mathbb{Q}[t^2, t^3]$ and $B = \mathbb{Q}[t]$. In this case, $\text{Pic}(A) \cong \mathbb{Q}$ and $\text{Pic}(B) = 0$.

Lemma 5.6. *If f is subintegral then $\mathcal{K}_{-q}^f = 0$ for $q > 0$.*

Proof. Each stalk of \mathcal{K}_{-q}^f is $K_{-q}(A, B)$, where $A \subset B$ is a subintegral extension of local rings. By Proposition 2.5 of [17], $K_{-q}(A, B) = 0$ for $q > 0$. Hence the result. \square

Proof of Theorem 1.3: We have a descent spectral sequence

$$E_2^{p,q} = H_{zar}^p(S, \mathcal{K}_{-q}^f) \Rightarrow K_{-p-q}(f).$$

By Lemma 5.6, $\mathcal{K}_{-q}^f = 0$ for $q > 0$. Moreover, $\mathcal{K}_0^f \cong f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ by the exact sequence (2.1) of [17]. Since the Zariski cohomological dimension is at most d , we get the first two assertions. The last assertion follows from Proposition 5.4(2). \square

Let X be a scheme. The seminormalization of X can be obtained by mimicking the normalization process. For each affine open subset $U = \text{Spec}(A)$ of X , let ${}^+A$ be the subintegral closure (or seminormalization) of A in its total quotient ring. Let ${}^+U = \text{Spec}({}^+A)$. Now by gluing together such schemes ${}^+U$ we get X_{sn} , which we call the seminormalization of X . Clearly, then $X_{\text{sn}} \rightarrow X$ is a subintegral morphism.

Corollary 5.7. *Let X be a d -dimensional noetherian scheme. Let X_{sn} be the seminormalization of X . Then $K_{-n}(X) \cong K_{-n}(X_{\text{sn}})$ for $n > d$ and $K_{-d}(X) \rightarrow K_{-d}(X_{\text{sn}})$ is surjective.*

Proof. Clear from Theorem 1.3(1) and the long exact sequence (2.1). \square

6. ON REGULARITY

A theorem of Vorst says that if a ring A is K_n -regular then it is K_{n-1} -regular (see V. 8.6 of [24]). Now we prove Theorem 1.4, which is a relative version of Vorst's result.

Proof of Theorem 1.4: First we suppose that $K_n(f) \cong K_n(f[s, t])$. Then $NK_n(f[s]) = 0$. The goal is to show that $NK_{n-1}(f) = 0$. Applying N to the exact sequence (2.1) for $f[s] : \text{Spec}(B[s]) \rightarrow \text{Spec}(A[s])$, we get the following exact sequence $\text{Seq}(NK_n, f[s])$

(6.1)

$$\cdots \rightarrow NK_n(f[s]) \rightarrow NK_n(A[s]) \rightarrow NK_n(B[s]) \rightarrow NK_{n-1}(f[s]) \rightarrow NK_{n-1}(A[s]) \rightarrow \cdots$$

By Theorem 5.6 of [17], $NK_n(f[s])$ is a $W(A[s])$ -module for all n . We know that $NK_n(A[s])$ is also a $W(A[s])$ -module (See IV.6.7 of [24]). Then $NK_n(B[s])$ is a $W(A[s])$ -module via the map $W(A[s]) \rightarrow W(B[s])$. In fact (6.1) is a sequence of $W(A[s])$ -modules. Consider the multiplicative closed set $T = \{[s]^q\}_{q \geq 0}$ in $W(A[s])$, where $[s]$ is the Teichmüller representative of s in $W(A[s])$. So after localization, we get the sequence $\text{Seq}((NK_n)_{[s]}, f[s])$, more precisely the terms are $NK_n(A[s])_{[s]}$, $NK_n(f[s])_{[s]}$ etc. Then we have a natural map $\text{Seq}((NK_n)_{[s]}, f[s]) \rightarrow \text{Seq}(NK_n, f[s, 1/s])$ of $T^{-1}W(A[s])$ -module. By Lemma V.8.5 of [24], $NK_n(A[s])_{[s]} \cong NK_n(A[s, 1/s])$. Now a diagram chase implies that $NK_n(f[s])_{[s]} \cong NK_n(f[s, 1/s])$. Applying the Bass fundamental theorem (see Example 5.1 of [17]) on $NK_n(f[s, 1/s])$, we get

$$NK_n(f[s, 1/s]) = NK_n(f) \oplus N^2K_n(f) \oplus N^2K_n(f) \oplus NK_{n-1}(f).$$

But $NK_n(f[s, 1/s]) = 0$ because $NK_n(f[s]) = 0$. Hence, $NK_{n-1}(f) = 0$.

Similarly, we can show that $NK_n(f[s, t])_{[t]} \cong NK_n(f[s, t, 1/t])$. Again by the Bass fundamental theorem, $N^2K_{n-1}(f) = 0$. Therefore, by repeating the same arguments we get $N^iK_{n-1}(f) = 0$ for all $i > 0$. \square

Proposition 6.1. *If f is a subintegral map of affine schemes then f can not be K_n -regular for $n \geq 0$. Moreover, $K_n(f) \neq 0$ for $n \geq 0$.*

Proof. Since f is subintegral, $K_0(f) \cong \text{Pic}(f)$ by Proposition 2.5 of [17]. Note that $NK_0(f) \cong NPic(f)$. By Theorem 1.5 of [15], $NPic(f) = 0$ if and only if f is seminormal. Therefore, $NK_0(f) \neq 0$ and hence f is not K_0 -regular. Now Theorem 1.4 implies that f can not be K_n -regular for $n \geq 0$.

Suppose $K_n(f) = 0$ for some $n \geq 0$. Since f is subintegral, so is $f[t_1, t_2, \dots, t_l]$. Then $K_n(f[t_1, \dots, t_l]) = 0$. This shows that f is K_n -regular, which is a contradiction by the first part. Hence, $K_n(f) \neq 0$ for $n \geq 0$. \square

Remark 6.2. The converse of Theorem 1.4 does not hold. Because, if f is a subintegral map of affine schemes then $K_n(f) = K_n(f[t_1, t_2, \dots, t_l]) = 0$ for $n < 0$ by Proposition 2.5

of [17]. Hence f is K_n -regular for $n < 0$. But f is not K_0 -regular by Proposition 6.1. In particular, consider $f : \operatorname{Spec}(\mathbb{Q}[x]/(x^2)) \rightarrow \operatorname{Spec}(\mathbb{Q})$. Here f is subintegral, K_{-1} -regular but not K_0 -regular.

7. RELATIVE VS ABSOLUTE

In this section, we prove Theorem 1.5. Recall that $KH(f) = \operatorname{hocolim}_j K(f \times \Delta^j)$. Then, there is a right half-plane spectral sequence (see Proposition 5.17 of [19]),

$$(7.1) \quad E_{p,q}^1 = K_q(f \times \Delta^p) \Rightarrow KH_{p+q}(f),$$

for any $f : X \rightarrow S$ map of schemes. This is the standard Bousfield-Kan spectral sequence of simplicial spectrum. If f is K_{-n} -regular for $n > d$, then the spectral sequence (7.1) implies that $K_{-n}(f) = KH_{-n}(f)$ for $n > d$.

Proof of Theorem 1.5:

(1) \Rightarrow (2) For a fix t , consider $f : \mathbb{A}_S^t \rightarrow S$. Then $K_{-n}(f) \cong KH_{-n}(f)$ for all $n > d$. Since KH is homotopy invariant, $KH_{-n}(f) = 0$ for $n > d$. Thus, $K_{-n}(f) = 0$ for $n > d$. Now by the exact sequence (2.1), $K_{-n}(S) \cong K_{-n}(\mathbb{A}_S^t)$ for $n \geq d$. We can argue for each t separately, hence S is K_{-n} -regular for $n \geq d$.

For the second assertion, consider the spectral sequence $K_q(S \times \Delta^p) \Rightarrow KH_{p+q}(S)$. By the first part S is K_{-n} -regular for $n \geq d$. Therefore, $K_{-n}(S) \cong KH_{-n}(S)$ for $n \geq d$. By Theorem 1 of [7], $KH_{-n}(S) = 0$ for $n > d$. Hence the result.

(2) \Rightarrow (1) This follows from Lemma 3.1 and Theorem 1.1. \square

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